CHAPTER III

CUTS

§ 1

Definition

Definition 28: *A set of rational numbers is called a cut if*

1) *it contains a rational number, but does not contain all rational numbers;*

2) *every rational number of the set is smaller than every rational number not belonging to the set;*

3) *it does not contain a greatest rational number* (i.e. a number which is greater than any other number of the set).

We will also use the term "lower class" for such a set, and the term "upper class" for the set of all rational numbers which are not contained in the lower class. The elements of the two sets will then be called "lower numbers" and "upper numbers," respectively.

Small Greek letters will be used throughout to denote cuts, except where otherwise specified.

Definition 29:

$$\xi = \eta$$

($=$ to be read "is equal to") *if every lower number for ξ is a lower number for η and every lower number for η is a lower number for ξ.*

In other words, if the sets are identical.

Otherwise,

$$\xi \neq \eta$$

($\neq$ to be read "is not equal to").

The following three theorems are trivial:

Theorem 116:

$$\xi = \xi.$$

Theorem 117: *If*

$$\xi = \eta$$

then

$$\eta = \xi.$$

Theorem 118: *If*

$$\xi = \eta,\ \eta = \zeta,$$

then

$$\xi = \zeta.$$

Theorem 119: *If X is an upper number for ξ and if*

$$X_1 > X,$$

then X_1 is an upper number for ξ.

Proof: Follows from 2) of Definition 28.

Theorem 120: *If X is a lower number for ξ and if*

$$X_1 < X,$$

then X_1 is a lower number for ξ.

Proof: Follows from 2) of Definition 28.

Conversely, the statement of Theorem 120 is of course equivalent to 2) of Definition 28. Thus if we wish to show that a given set of rational numbers is a cut, we need show only the following:

1) The set is not empty, and there is a rational number not belonging to it.

2) With every number it contains, the set also contains all numbers smaller than that number.

3) With every number it contains, the set also contains a greater one.

§ 2

Ordering

Definition 30: *If ξ and η are cuts, then*

$$\xi > \eta$$

($>$ to be read "is greater than") *if there exists a lower number for ξ which is an upper number for η.*

Definition 31: *If ξ and η are cuts, then*

$$\xi < \eta$$

($<$ to be read "is less than") *if there exists an upper number for ξ which is a lower number for η.*

Theorem 121: *If*

$$\xi > \eta$$

then

$$\eta < \xi.$$

Proof: Each means that there exists an upper number for η which is a lower number for ξ.

Theorem 122: *If*

$$\xi < \eta$$

then

$$\eta > \xi.$$

Proof: Each means that there exists a lower number for η which is an upper number for ξ.

Theorem 123: *For any given ξ, η, exactly one of*

$$\xi = \eta,\ \xi > \eta,\ \xi < \eta$$

is the case.

Proof: 1)

$$\xi = \eta,\ \xi > \eta$$

are incompatible by Definition 29 and Definition 30.

$$\xi = \eta,\ \xi < \eta$$

are incompatible by Definition 29 and Definition 31.

If we had

$$\xi > \eta,\ \xi < \eta,$$

it would follow that there exists a lower number X for ξ which is an upper number for η, and that there also exists an upper number Y for ξ which is a lower number for η. By 2) of Definition 28, we would then have both

$$X < Y \text{ and } X > Y.$$

Therefore we can have at most one of the three cases.

2) If

$$\xi \neq \eta,$$

then the lower classes do not coincide. Then we either have that some lower number for ξ is an upper number for η, in which case it follows that

$$\xi > \eta;$$

or we have that some lower number for η is an upper number for ξ, in which case it follows that

$$\xi < \eta.$$

Definition 32: $\xi \geqq \eta$

means

$$\xi > \eta \text{ or } \xi = \eta.$$

($\geqq$ to be read "is greater than or equal to.")

Definition 33: $\xi \leqq \eta$

means

$$\xi < \eta \text{ or } \xi = \eta.$$

($\leqq$ to be read "is less than or equal to.")

Theorem 124: *If*

$$\xi \geqq \eta$$

then

$$\eta \leqq \xi.$$

Proof: Theorem 121.

Theorem 125: *If*

$$\xi \leqq \eta$$

then

$$\eta \geqq \xi.$$

Proof: Theorem 122.

Theorem 126 (Transitivity of Ordering): *If*

$$\xi < \eta,\ \eta < \zeta,$$

then

$$\xi < \zeta.$$

Proof: There exists an upper number X for ξ which is a lower number for η; there also exists an upper number Y for η which is a lower number for ζ. Applying property 2) of cuts (cf. Definition 28) to the cut η, we obtain

$$X < Y,$$

so that Y is an upper number for ξ. Therefore

$$\xi < \zeta.$$

Theorem 127: *If*

$$\xi \leqq \eta,\ \eta < \zeta \ \textit{or}\ \xi < \eta,\ \eta \leqq \zeta,$$

then

$$\xi < \zeta.$$

Proof: Obvious if the equality sign holds in the hypothesis; otherwise, Theorem 126 does it.

Theorem 128: *If*

$$\xi \leqq \eta,\ \eta \leqq \zeta,$$

then

$$\xi \leqq \zeta.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise, Theorem 127 does it.

§ 3

Addition

Theorem 129: I) *Let ξ and η be cuts. Then the set of all rational numbers which are representable in the form $X + Y$, where X is a lower number for ξ and Y is a lower number for η, is itself a cut.*

II) *No number of this set can be written as a sum of an upper number for ξ and an upper number for η.*

Proof: 1) Consider any lower number X for ξ and any lower number Y for η. Then $X + Y$ belongs to our set.

Next, consider any upper number X_1 for ξ and any upper number Y_1 for η; if X and Y are any lower numbers for ξ and for η respectively, we have

$$X < X_1, \quad Y < Y_1,$$

hence

$$X + Y < X_1 + Y_1,$$
$$X_1 + Y_1 \neq X + Y;$$

therefore $X_1 + Y_1$ does not belong to our set, and we have proved II), as well as property 1) of cuts for our set.

2) To prove that our set satisfies property 2) of cuts, we must show that with any number it contains, our set also contains all numbers which are less than that number. Let

$$Z < X + Y,$$

where X and Y are lower numbers for ξ and η respectively. Then

$$(X + Y)\frac{Z}{X+Y} < (X+Y)\cdot 1,$$

hence, by Theorem 106,

$$\frac{Z}{X+Y} < 1,$$

so that, by Theorem 105,

$$X\frac{Z}{X+Y} < X\cdot 1 = X$$

and

$$Y\frac{Z}{X+Y} < Y\cdot 1 = Y;$$

therefore by virtue of the second property of cuts as applied to ξ and η, the numbers $X\frac{Z}{X+Y}$ and $Y\frac{Z}{X+Y}$ are lower numbers for ξ and η respectively.

The sum of those two rational numbers is the given Z, since

$$X\frac{Z}{X+Y}+Y\frac{Z}{X+Y}=(X+Y)\frac{Z}{X+Y}=Z.$$

3) Any given number of our set is of the form $X + Y$ where X and Y are lower numbers for ξ and η respectively. Using the third property of cuts as applied to ξ, we can find a lower number

$$X_1 > X$$

for ξ; then

$$X_1 + Y > X + Y,$$

so that there exists in our set a number which is $> X + Y$.

Definition 34: *The cut constructed in Theorem* 129 *is denoted by* $\xi + \eta$ (+ to be read "plus") *and is called the sum of* ξ *and* η, *or the cut obtained from addition of* η *to* ξ.

Theorem 130 (Commutative Law of Addition):

$$\xi + \eta = \eta + \xi.$$

Proof: Every $X + Y$ is a $Y + X$, and vice versa.

Theorem 131 (Associative Law of Addition):

$$(\xi + \eta) + \zeta = \xi + (\eta + \zeta).$$

Proof: Every $(X + Y) + Z$ is an $X + (Y + Z)$, and vice versa.

Theorem 132: *Given any* A, *and given a cut, then there exist a lower number* X *and an upper number* U *for the cut such that*

$$U - X = A.$$

Proof: Let X_1 be some lower number, and consider all rational numbers

$$X_1 + nA$$

where n is an integer. Not all of these are lower numbers; for if Y is any upper number, then

$$Y > X_1;$$

hence by Theorem 115, we have for some suitable n that

$$nA > Y - X_1,$$

$$X_1 + nA > (Y - X_1) + X_1 = Y,$$

so that $X_1 + nA$ is an upper number.

The set of all n for which $X_1 + nA$ is an upper number contains a smallest integer, by Theorem 27; we will denote it by u.

If

$$u = 1,$$

then we set

$$X = X_1, \ U = X_1 + A;$$

if

$$u > 1,$$

then we set

$$X = X_1 + (u - 1)A, \ U = X_1 + uA = X + A.$$

In each case, X is a lower and U an upper number, and

$$U - X = A.$$

Theorem 133: $\xi + \eta > \xi.$

Proof: Let Y be a lower number for η. By Theorem 132, we can find a lower number X for ξ and an upper number U for ξ such that

$$U - X = Y;$$

then the number

$$U = X + Y$$

is an upper number for ξ and a lower number for $\xi + \eta$. Therefore

$$\xi + \eta > \xi.$$

Theorem 134: *If* $\xi > \eta$ *then*

$$\xi + \zeta > \eta + \zeta.$$

Proof: There exists an upper number Y for η which is a lower number for ξ. Choose a greater lower number

$$X > Y$$

for ξ; it is an upper number for η. Now by Theorem 132, we can find an upper number Z for ζ and a lower number U for ζ such that

$$Z - U = X - Y.$$

Then we have

$$Y + Z = Y + ((X - Y) + U) = (Y + (X - Y)) + U = X + U,$$

so that this number, besides being a lower number for $\xi + \zeta$, is also (by Theorem 129, II)) an upper number for $\eta + \zeta$. Therefore

$$\xi + \zeta > \eta + \zeta.$$

Theorem 135: *If*

$$\xi > \eta, \ \textit{or} \ \xi = \eta, \ \textit{or} \ \xi < \eta,$$

then

$$\xi + \zeta > \eta + \zeta, \text{ or } \xi + \zeta = \eta + \zeta, \text{ or } \xi + \zeta < \eta + \zeta,$$

respectively.

Proof: The first part is Theorem 134, the second is obvious, and the third follows from the first since

$$\eta > \xi,$$
$$\eta + \zeta > \xi + \zeta,$$
$$\xi + \zeta < \eta + \zeta.$$

Theorem 136: *If*

$$\xi + \zeta > \eta + \zeta, \text{ or } \xi + \zeta = \eta + \zeta, \text{ or } \xi + \zeta < \eta + \zeta,$$

then

$$\xi > \eta, \text{ or } \xi = \eta, \text{ or } \xi < \eta, \text{ respectively.}$$

Proof: Follows from Theorem 135, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 137: *If*

$$\xi > \eta, \quad \zeta > v$$

then

$$\xi + \zeta > \eta + v.$$

Proof: By Theorem 134, we have

$$\xi + \zeta > \eta + \zeta$$

and

$$\eta + \zeta = \zeta + \eta > v + \eta = \eta + v,$$

so that

$$\xi + \zeta > \eta + v.$$

Theorem 138: *If*

$$\xi \geqq \eta, \ \zeta > v \text{ or } \xi > \eta, \ \zeta \geqq v$$

then

$$\xi + \zeta > \eta + v.$$

Proof: Follows from Theorem 134 if the equality sign holds in the hypothesis; otherwise from Theorem 137.

Theorem 139: *If*

$$\xi \geqq \eta, \quad \zeta \geqq v$$

then

$$\xi + \zeta \geqq \eta + v.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise, Theorem 138 does it.

Theorem 140: *If*

$$\xi > \eta,$$

then

$$\eta + v = \xi$$

has exactly one solution v.

Preliminary Remark: If

$$\xi \leqq \eta,$$

then there does not exist a solution, by Theorem 133.

Proof: I) There exists at most one solution; for if

$$v_1 \neq v_2$$

then, by Theorem 135,

$$\eta + v_1 \neq \eta + v_2.$$

II) I will show first that the set of all rational numbers of the form $X - Y$ (hence $X > Y$) where X is a lower number for ξ and Y is an upper number for η, constitutes a cut.

1) We know from the first part of the proof of Theorem 134 that such an $X - Y$ does indeed exist.

No upper number X_1 for ξ can constitute such an $X - Y$, since each number of this form satisfies

$$X - Y < (X - Y) + Y = X < X_1.$$

2) If an $X - Y$ of the above sort is given and if

$$U < X - Y,$$

then

$$U + Y < (X - Y) + Y = X,$$

so that the number

$$U + Y = X_2$$

is a lower number for ξ, and the number

$$U = X_2 - Y$$

belongs to our set.

3) If an $X - Y$ of the above sort is given, choose a lower number

$$X_3 > X$$

for ξ. Then

$$(X_3 - Y) + Y > (X - Y) + Y,$$
$$X_3 - Y > X - Y,$$

so that $X_3 - Y$ is a number of our set which is greater than the given number $X - Y$.

Our set is therefore a cut; let us denote it by v.

We will show that it satisfies

$$\eta + \upsilon = \xi.$$

To prove this, it suffices to establish the following two statements:

A) Every lower number for $\upsilon + \eta$ is a lower number for ξ.

B) Every lower number for ξ is a lower number for $\upsilon + \eta$.

As regards A): Every lower number for $\upsilon + \eta$ is of the form

$$(X - Y) + Y_1$$

where X is a lower number for ξ, Y an upper number for η, Y_1 a lower number for η, and

$$X > Y.$$

Now we have

$$Y > Y_1,$$

$$((X-Y)+Y_1)+(Y-Y_1) = (X-Y)+(Y_1+(Y-Y_1)) = (X-Y)+Y = X,$$

$$(X-Y)+Y_1 < X,$$

so that $(X - Y) + Y_1$ is a lower number for ξ.

As regards B): a) Let the given lower number for ξ be at the same time an upper number for η, and denote it by Y. Choose a lower number X for ξ such that

$$X > Y,$$

and moreover choose, by Theorem 132, a lower number Y_1 for η and an upper number Y_2 for η such that

$$Y_2 - Y_1 = X - Y.$$

Then we have

$$Y > Y_1,$$

hence

$$Y_2 + (Y - Y_1) = ((X-Y)+Y_1)+(Y-Y_1) = (X-Y)+(Y_1+(Y-Y_1))$$
$$= (X-Y)+Y = X,$$

$$Y - Y_1 = X - Y_2,$$

$$Y = (Y-Y_1)+Y_1 = (X-Y_2)+Y_1,$$

so that Y is a lower number for $\upsilon + \eta$.

b) If the given lower number for ξ is also a lower number for η, then it is less than all those rational numbers which were considered in a) and which turned out to be lower numbers for $\upsilon + \eta$. Hence in this case the given number must itself be a lower number for $\upsilon + \eta$.

Definition 35: *The* υ *of Theorem* 140 *is denoted by* $\xi - \eta$ (— to be read "minus") *and is called the difference* ξ *minus* η, *or the cut obtained by subtraction of* η *from* ξ.

§ 4

Multiplication

Theorem 141: I) *Let ξ and η be cuts. Then the set of all rational numbers which are representable in the form XY, where X is a lower number for ξ and Y is a lower number for η, is itself a cut.*

II) *No number of this set can be written as a product of an upper number for ξ and an upper number for η.*

Proof: 1) Consider any lower number X for ξ and any lower number Y for η; then XY belongs to the set.

Next, consider any upper number X_1 for ξ and any upper number Y_1 for η. If X and Y are any lower numbers for ξ and for η respectively, we have

$$X < X_1, \quad Y < Y_1,$$

hence

$$XY < X_1 Y_1,$$
$$X_1 Y_1 \neq XY;$$

therefore X_1Y_1 does not belong to our set, and we have proved II), as well as property 1) of cuts, for our set.

2) Let X be a lower number for ξ, Y a lower number for η, and let

$$Z < XY.$$

Then we have

$$X\left(\frac{1}{X}Z\right) = \left(X\frac{1}{X}\right)Z = 1 \cdot Z = Z,$$

$$\frac{Z}{X} = \frac{1}{X}Z < \frac{1}{X}(XY) = \left(\frac{1}{X}X\right)Y = Y,$$

so that $\frac{Z}{X}$ is a lower number for η. The equation

$$Z = X\frac{Z}{X}$$

thus shows that Z belongs to our set.

3) Let there be given any number of the set; it is of the form XY where X and Y are lower numbers for ξ and for η respectively. Choose a lower number

$$X_1 > X$$

for ξ; then we have that

$$X_1 Y > XY,$$

so that our set contains a number which is $> XY$.

Definition 36: *The cut constructed in Theorem* 141 *is denoted by* $\xi \cdot \eta$ ($\cdot$ to be read "times"; however the dot is usually omitted), *and is called the product of* ξ *and* η, *or the cut obtained from multiplication of* ξ *by* η.

Theorem 142 (Commutative Law of Multiplication):

$$\xi\eta = \eta\xi.$$

Proof: Every XY is a YX, and vice versa.

Theorem 143 (Associative Law of Multiplication):

$$(\xi\eta)\zeta = \xi(\eta\zeta).$$

Proof: Every $(XY)Z$ is an $X(YZ)$, and vice versa.

Theorem 144 (Distributive Law):

$$\xi(\eta + \zeta) = \xi\eta + \xi\zeta.$$

Proof: I) Every lower number for $\xi(\eta + \zeta)$ is of the form

$$X(Y + Z) = XY + XZ$$

where X, Y and Z are lower numbers for ξ, η, and ζ, respectively. The number $XY + XZ$ is a lower number for $\xi\eta + \xi\zeta$.

II) Every lower number for $\xi\eta + \xi\zeta$ is of the form

$$XY + X_1 Z$$

where X, Y, X_1, and Z are lower numbers for ξ, η, ξ, and ζ, respectively. Let X_2 stand for the number X in case $X \geqq X_1$ and for the number X_1 in case $X < X_1$; then X_2 is a lower number for ξ, so that $X_2(Y + Z)$ is a lower number for $\xi(\eta + \zeta)$. From

$$XY \leqq X_2 Y,$$
$$X_1 Z \leqq X_2 Z$$

follows

$$XY + X_1 Z \leqq X_2 Y + X_2 Z = X_2(Y + Z);$$

hence $XY + X_1 Z$ is a lower number for $\xi(\eta + \zeta)$.

Theorem 145: *If*

$$\xi > \eta, \text{ or } \xi = \eta, \text{ or } \xi < \eta,$$

then

$$\xi\zeta > \eta\zeta, \text{ or } \xi\zeta = \eta\zeta, \text{ or } \xi\zeta < \eta\zeta, \text{ respectively.}$$

Proof: 1) If $\qquad \xi > \eta,$

then we have by Theorem 140 that, with a suitable v,

$$\xi = \eta + v,$$

hence

$$\xi\zeta = (\eta + v)\zeta = \eta\zeta + v\zeta > \eta\zeta.$$

2) If

$$\xi = \eta$$

then obviously

$$\xi\zeta = \eta\zeta.$$

3) If

$$\xi < \eta$$

then

$$\eta > \xi,$$

so that by 1),

$$\eta\zeta > \xi\zeta,$$
$$\xi\zeta < \eta\zeta.$$

Theorem 146: *If*

$$\xi\zeta > \eta\zeta, \text{ or } \xi\zeta = \eta\zeta, \text{ or } \xi\zeta < \eta\zeta,$$

then

$$\xi > \eta, \text{ or } \xi = \eta, \text{ or } \xi < \eta, \text{ respectively.}$$

Proof: Follows from Theorem 145, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 147: *If*

$$\xi > \eta, \ \zeta > v$$

then

$$\xi\zeta > \eta v.$$

Proof: By Theorem 145,

$$\xi\zeta > \eta\zeta$$

and

$$\eta\zeta = \zeta\eta > v\eta = \eta v,$$

so that

$$\xi\zeta > \eta v.$$

Theorem 148: *If*

$$\xi \geqq \eta, \ \zeta > v \text{ or } \xi > \eta, \ \zeta \geqq v,$$

then

$$\xi\zeta > \eta v.$$

Proof: Follows from Theorem 145 if an equality sign holds in the hypothesis; otherwise from Theorem 147.

Theorem 149: *If*

$$\xi \geqq \eta, \ \zeta \geqq v,$$

then

$$\xi\zeta \geqq \eta v.$$

Proof: Obvious if two equality signs hold in the hypothesis; otherwise, Theorem 148 does it.

Theorem 150: *For any given rational number R, the set of all rational numbers < R constitutes a cut.*

Proof: 1) By Theorem 90, there does exist an $X < R$. The number R itself is not $< R$.

2) If

$$X < R,\ X_1 \geqq R,$$

then

$$X < X_1.$$

3) If

$$X < R,$$

then by Theorem 91 there exists an X_1 such that

$$X < X_1 < R.$$

Definition 37: *The cut constructed in Theorem* 150 *is denoted by* R^*.

(Thus capital italic letters with asterisks will stand for cuts, not for rational numbers.)

Theorem 151: $$\xi \cdot 1^* = \xi.$$

Proof: $\xi \cdot 1^*$ is the set of all XY where X is a lower number for ξ and

$$Y < 1.$$

Every such XY is $< X$ and thus is a lower number for ξ.

Conversely, let there be given a lower number X for ξ. Choose, for ξ, a lower number

$$X_1 > X$$

and set

$$Y = \frac{1}{X_1}X.$$

Then

$$Y < \frac{1}{X_1}X_1 = 1,$$

so that the number

$$X = X_1 Y$$

is a lower number for $\xi \cdot 1^*$.

Theorem 152: *For any given* ξ, *the equation*

$$\xi v = 1^*$$

has a solution v.

Proof: Consider the set of all numbers $\frac{1}{X}$ where X may be any upper number for ξ, excepting only the least upper number (if such a one exists). We will show that this set is a cut.

1) The set does contain a number; for if X is an upper number for ξ, then so is $X+X$, and the latter indeed can not be the smallest, so that $\frac{1}{X+X}$ belongs to our set.

There exists a rational number which does not belong to the set; for if X_1 is any lower number for ξ, then any upper number X for ξ satisfies

$$X \neq X_1,$$

hence, since

$$X\frac{1}{X} = 1 = X_1\frac{1}{X_1},$$

$$\frac{1}{X} \neq \frac{1}{X_1};$$

thus $\frac{1}{X_1}$ does not belong to our set.

2) Consider any number $\frac{1}{X}$ of our set; then X is an upper number for ξ. Now if

$$U < \frac{1}{X},$$

then

$$UX < \frac{1}{X}X = 1 = U\frac{1}{U},$$

hence

$$X < \frac{1}{U},$$

so that $\frac{1}{U}$ is an upper number for ξ, and is not the least such. Since

$$U\frac{1}{U} = 1,$$

$$U = \frac{1}{\frac{1}{U}},$$

the number U belongs to our set.

3) Let there be given a number $\frac{1}{X}$ of our set; then X is an upper number for ξ, and is not the least such. Choose an upper number

$$X_1 < X$$

for ξ, and choose (Theorem 91) an X_2 such that

$$X_1 < X_2 < X.$$

Then X_2 is an upper number for ξ, and is not the least such. From

$$X_2 \frac{1}{X} < X \frac{1}{X} = 1 = X_2 \frac{1}{X_2}$$

we obtain

$$\frac{1}{X_2} > \frac{1}{X},$$

so that we have found a number in our set which is greater than the given one.

Our set is therefore a cut; let it be denoted by v.

We will show that it satisfies.

$$\xi v = 1^*.$$

To prove this, it suffices to establish the following two statements:

A) Every lower number for ξv is < 1.

B) Every rational number < 1 is a lower number for ξv.

As regards A): Every lower number for ξv is of the form

$$X \frac{1}{X_1},$$

where X is a lower number for ξ and X_1 an upper number for ξ. Now

$$X < X_1$$

implies

$$X \frac{1}{X_1} < X_1 \frac{1}{X_1} = 1.$$

As regards B): Let

$$U < 1.$$

Choose any lower number X for ξ and then, by Theorem 132, a lower number X_1 for ξ and an upper number X_2 for ξ such that

$$X_2 - X_1 = (1 - U)X.$$

Then we have

$$X_2 - X_1 < (1 - U)X_2,$$

$$(X_2 - X_1) + UX_2 < (1 - U)X_2 + UX_2 = X_2 = (X_2 - X_1) + X_1,$$

$$UX_2 < X_1,$$

$$X_2 = \left(\frac{1}{U} U\right) X_2 = \frac{1}{U}(UX_2) < \frac{1}{U} X_1 = \frac{X_1}{U}.$$

Therefore $\frac{X_1}{U}$ is an upper number for ξ, and is not the least such.

If

$$U\frac{X_1}{U} = X_1$$

then

$$U = \frac{X_1}{\frac{X_1}{U}} = X_1 \frac{1}{\frac{X_1}{U}};$$

here, X_1 is a lower number for ξ, and $\frac{1}{\frac{X_1}{U}}$ is a lower number for v; hence U is a lower number for ξv.

Theorem 153: *The equation*

$$\eta v = \xi,$$

where ξ and η are given, has exactly one solution v.

Proof: I) There exists at most one solution; for if

$$v_1 \neq v_2$$

then, by Theorem 145,

$$\eta v_1 \neq \eta v_2.$$

II) If τ is the solution—whose existence is proved by Theorem 152—of the equation

$$\eta \tau = 1^*,$$

then the cut

$$v = \tau \xi$$

satisfies the equation in Theorem 153; for we have, by Theorem 151, that

$$\eta v = \eta(\tau\xi) = (\eta\tau)\xi = 1^*\xi = \xi.$$

Definition 38: *The v of Theorem* 153 *is denoted by* $\frac{\xi}{\eta}$ (to be read "ξ over η"), *and is called the quotient of ξ by η, or the cut obtained from division of ξ by η.*

§ 5

Rational Cuts and Integral Cuts

Definition 39: *A cut of the form* X^* *is called a rational cut.*

Definition 40: *A cut of the form* x^* *is called an integral cut.*

(Thus small italic letters with asterisks stand for cuts, not for integers.)

Theorem 154: *If*

$$X > Y, \text{ or } X = Y, \text{ or } X < Y,$$

then

$$X^* > Y^*, \text{ or } X^* = Y^*, \text{ or } X^* < Y^*, \text{ respectively,}$$

and vice versa.

Proof: I) 1) If

$$X > Y,$$

then Y is a lower number for X^*. The number Y is an upper number for Y^*. Therefore

$$X^* > Y^*.$$

2) If

$$X = Y$$

then clearly

$$X^* = Y^*.$$

3) If

$$X < Y$$

then

$$Y > X,$$

hence, by 1),

$$Y^* > X^*,$$

$$X^* < Y^*.$$

II) The converse is obvious, since the three cases are, in both instances, mutually exclusive and exhaust all possibilities.

Theorem 155: $(X+Y)^* = X^* + Y^*;$

$$(X-Y)^* = X^* - Y^*, \quad \text{if } X > Y;$$

$$(XY)^* = X^* Y^*;$$

$$\left(\frac{X}{Y}\right)^* = \frac{X^*}{Y^*}.$$

Proof: I) a) Every lower number for $X^* + Y^*$ is the sum of a rational number $< X$ and of a rational number $< Y$; it is therefore $< X + Y$, and is thus a lower number for $(X + Y)^*$.

b) Every lower number U for $(X + Y)^*$ is $< X + Y$. Now

$$\frac{U}{X+Y} < 1,$$

$$U = X\frac{U}{X+Y} + Y\frac{U}{X+Y}$$

implies that U, as the sum of a rational number $< X$ and of a rational number $< Y$, is a lower number for $X^* + Y^*$.

Hence we have

$$(X+Y)^* = X^* + Y^*.$$

II) If

$$X > Y$$

then

$$X = (X - Y) + Y,$$

so that by 1),

$$X^* = (X-Y)^* + Y^*,$$

$$(X-Y)^* = X^* - Y^*.$$

III) a) Every lower number for X^*Y^* is the product of a rational number $< X$ and of a rational number $< Y$; therefore it is $< XY$, and so is a lower number for $(XY)^*$.

b) Every lower number U for $(XY)^*$ is $< XY$. Choose a rational number U_1, by Theorem 91, such that

$$U < U_1 < XY$$

Then

$$\frac{U}{U_1} < 1$$

and

$$\frac{U_1}{Y} < X.$$

Thus the relation

$$U = \frac{U_1}{Y}\left(Y\frac{U}{U_1}\right)$$

represents U as the product of a lower number for X^* and a lower number for Y^*. Therefore U is a lower number for X^*Y^*.

Hence we have

$$(XY)^* = X^*Y^*.$$

IV)

$$X = \frac{X}{Y}Y,$$

so that by III), we obtain

$$X^* = \left(\frac{X}{Y}\right)^* Y^*,$$

$$\left(\frac{X}{Y}\right)^* = \frac{X^*}{Y^*}.$$

Theorem 156: *The integral cuts satisfy the five axioms of the natural numbers if the role of 1 is assigned to 1^* and if we set*

$$(x^*)' = (x')^*.$$

Proof: Let $\mathfrak{Z}^*$ be the set of all integral cuts.

1) 1^* belongs to $\mathfrak{Z}^*$.

2) For every x^* in $\mathfrak{Z}^*$, the cut $(x^*)'$ is also in $\mathfrak{Z}^*$.

3) We always have

$$x' \neq 1,$$

hence

$$(x')^* \neq 1^*,$$
$$(x^*)' \neq 1^*.$$

4) If

$$(x^*)' = (y^*)'$$

then

$$(x')^* = (y')^*,$$
$$x' = y',$$
$$x = y,$$
$$x^* = y^*.$$

5) Let a set $\mathfrak{M}^*$ of integral cuts have the following properties:

I) 1^* belongs to $\mathfrak{M}^*$.

II) If x^* belongs to $\mathfrak{M}^*$, then so does $(x^*)'$.

Also, denote by $\mathfrak{M}$ the set of x for which x^* belongs to $\mathfrak{M}^*$. Then 1 belongs to $\mathfrak{M}$, and if x belongs to $\mathfrak{M}$ then so does x'. Hence every integer belongs to $\mathfrak{M}$, so that every integral cut belongs to $\mathfrak{M}^*$.

Since $=$, $>$, $<$, sum, difference (whenever it exists), product, and quotient, in the domain of rational cuts all correspond to the earlier concepts (by Theorems 154 and 155), the rational cuts have all the properties which we have proved, in Chapter 2, attach to the rational numbers; the integral cuts, in particular, have all the properties that have been established for the integers.

Therefore, we throw out the rational numbers and replace them by the corresponding rational cuts, so that in all that follows we will only have to speak in terms of cuts whenever any of the foregoing material is involved. (However, the rational numbers survive—in sets—in the concept of cut.)

Definition 41: *The symbol X* (now freed of its previous meaning) *will denote the rational cut X^* to which we also transfer the name "rational number"; the name "integer" will similarly be transferred, to apply to integral cuts.*

Thus, for instance, we will simply write

$$\xi \frac{1}{\xi} = 1$$

instead of

$$\xi \frac{1^*}{\xi} = 1^*.$$

Theorem 157: *The rational numbers are those cuts for which there exists a least upper number X. This X is then the cut.*

Proof: 1) For the cut X (our old X^*), X (the rational number in the old sense) is a least upper number.

2) If there exists a least upper number X for a cut ξ, then every lower number for ξ is $< X$ and every upper number is $\geqq X$, so that the cut is X (the old X^*).

Theorem 158: *Let ξ be a cut. Then X is a lower number if, and only if,*

$$X < \xi,$$

and hence is an upper number if, and only if,

$$X \geqq \xi.$$

Proof: 1) If X is a lower number for ξ, then, noting that X is an upper number for X (the old X^*), we have

$$X < \xi.$$

2) If X is an upper number for ξ, and is the least such, then we have by Theorem 157 that

$$X = \xi.$$

3) If X is an upper number for ξ but is not the least such, we choose an upper number X_1 less than X. Then X_1 is a lower number for X, so that

$$X > \xi.$$

Theorem 159: *If*

$$\xi < \eta,$$

then there exists a Z such that

$$\xi < Z < \eta.$$

Proof: Choose an upper number X for ξ which is a lower number for η, and then choose a greater lower number Z for η. Then we have by Theorem 158 that

$$\xi \leqq X < Z < \eta.$$

Theorem 160: *Every*

$$Z > \xi\eta$$

may be brought into the form

$$Z = XY, \ X \geqq \xi, \ Y \geqq \eta.$$

Proof: Denote by ζ the lesser of the two cuts 1 and $\frac{Z - \xi\eta}{(\xi+\eta)+1}$.

Then

$$\zeta \leqq 1, \ \zeta \leqq \frac{Z - \xi\eta}{(\xi+\eta)+1}.$$

Choose Z_1 and Z_2 by Theorem 159 such that

$$\xi < Z_1 < \xi + \zeta, \ \eta < Z_2 < \eta + \zeta.$$

Then we have

$$Z_1 Z_2 < (\xi+\zeta)(\eta+\zeta) = (\xi+\zeta)\eta + (\xi+\zeta)\zeta \leqq (\xi+\zeta)\eta + (\xi+1)\zeta$$
$$= (\xi\eta + \eta\zeta) + (\xi+1)\zeta = \xi\eta + ((\xi+\eta)+1)\zeta \leqq \xi\eta + (Z - \xi\eta) = Z.$$

By means of

$$Z = \frac{Z}{Z_2} Z_2$$

and using

$$X = \frac{Z}{Z_2} = Z\frac{1}{Z_2} > (Z_1 Z_2)\frac{1}{Z_2} = Z_1 > \xi,$$
$$Y = Z_2 > \eta,$$

we have decomposed Z as required.

Theorem 161: *For each ζ, the equation*

$$\xi\xi = \zeta$$

has exactly one solution.

Proof: I) There exists at most one solution; for if

$$\xi_1 > \xi_2$$

then

$$\xi_1 \xi_1 > \xi_2 \xi_2.$$

II) Consider the set of all rational numbers X for which

$$XX < \zeta.$$

This set constitutes a cut, for:

1) If

$$X < 1 \text{ and } X < \zeta,$$

then

$$XX < X \cdot 1 = X < \zeta.$$

If

$$X \geqq 1 \text{ and } X \geqq \zeta,$$

then

$$XX \geqq X \cdot 1 = X \geqq \zeta.$$

2) If

$$XX < \zeta, \; Y < X$$

then

$$YY < XX < \zeta.$$

3) Let

$$XX < \zeta.$$

Choose a Z less than the lesser of the two cuts 1 and $\dfrac{\zeta - XX}{X + (X+1)}$. Then

$$Z < 1, \; Z < \frac{\zeta - XX}{X + (X+1)};$$

furthermore, we have

$$X + Z > X$$

and

$$(X+Z)(X+Z) = (X+Z)X + (X+Z)Z < (XX + ZX) + (X+1)Z$$
$$= XX + (X + (X+1))Z < XX + (\zeta - XX) = \zeta.$$

If we denote by ξ the cut which we have constructed, then we now assert that

$$\xi\xi = \zeta.$$

If we had

$$\xi\xi > \zeta,$$

then we could choose, by Theorem 159, a Z such that

$$\xi\xi > Z > \zeta.$$

This Z, being a lower number for $\xi\xi$, would satisfy

$$Z = X_1 X_2, \; X_1 < \xi, \; X_2 < \xi;$$

if X denotes the greater of the two numbers X_1 and X_2, then we would have, in contradiction to the above, that

$$X < \xi,$$
$$Z \leqq XX < \zeta.$$

If we had

$$\xi\xi < \zeta,$$

then we could choose, by Theorem 159, a Z such that

$$\xi\xi < Z < \zeta.$$

By Theorem 160, Z would be of the form

$$Z = X_1 X_2,\ X_1 \geqq \xi,\ X_2 \geqq \xi;$$

if X denotes the lesser of the two numbers X_1 and X_2, then we would have, in contradiction to the above, that

$$X \geqq \xi,$$
$$Z \geqq XX \geqq \zeta.$$

Definition 42: *Any cut which is not a rational number is called an irrational number.*

Theorem 162: *There exists an irrational number.*

Proof: It suffices to show that the solution of

$$\xi\xi = 1',$$

whose existence is guaranteed by Theorem 161, is irrational.

Otherwise, we would have

$$\xi = \frac{x}{y};$$

among all such representations we choose one, by Theorem 27, for which y is as small as possible. Since

$$1' = \xi\xi = \frac{x}{y}\cdot\frac{x}{y} = \frac{xx}{yy},$$

we have

$$yy < 1'(yy) = xx = (1'y)y < (1'y)(1'y),$$
$$y < x < 1'y.$$

Set

$$x - y = u.$$

Then

$$y + u = x < 1'y = y + y,$$
$$u < y.$$

Now we have that

$$(v+w)(v+w) = (v+w)v + (v+w)w = (vv+wv) + (vw+ww)$$
$$= (vv + 1'(vw)) + ww,$$

hence, setting

that
$$y-u = t$$

$$\begin{aligned} xx+tt &= (y+u)(y+u)+tt = (yy+1'(yu))+(uu+tt) \\ &= (yy+(1'u)(u+t))+(uu+tt) \\ &= (yy+1'(uu))+((1'(ut)+uu)+tt) \\ &= (yy+1'(uu))+(u+t)(u+t) \\ &= (yy+1'(uu))+yy = 1'(yy)+1'(uu) = xx+1'(uu), \end{aligned}$$

$$tt = 1'(uu),$$

$$\frac{t}{u}\cdot\frac{t}{u} = 1',$$

which contradicts

$$u < y.$$