Baby Rudin supplementary problems, week 1.

1. The claim is that $-(x+y) = (-x)+(-y)$:–

$$(x+y)+((-x)+(-y)) = x+(-x)+y+(-y) \quad \text{(associativity, commutativity)}$$
$$= 0+0$$
$$= 0.$$

So $(-x)+(-y)$ is an additive inverse for $(x+y)$.

2. Let S be a set; $a \in S$ is the minimum of S if for all $s \in S$, $s \geq a$.

~~Suppose~~ $-S = \{-s : s \in S\}$ has a minimum, a
$\Leftrightarrow \exists a \in -S . \forall x \in -S . \; x \geq a$
$\Leftrightarrow \exists a' \in S . \; \forall x' \in S . \; (-x') \geq (-a')$ thm. 1.18c
$\Leftrightarrow \exists a' \in S . \; \forall x' \in S . \; x' \leq a'$.
$\Leftrightarrow S$ has a maximum, a' $(= -a)$.

If S has a max, then that max is a sup of S. Similarly, if S has a min, then that min is an inf of S. Formally, we have $(\sup S \text{ exists}) \wedge (\sup S \in S) \Rightarrow \sup S = \max S$.
Note: $\exists$ sets with ~~max~~ sup but not max (e.g. $\{-1/n : n \in \mathbb{N}\}$.

3. A : no sup (if $a = \sup A$, then $a = 1/x + x$ for some x; and $\frac{1}{x+1} + x + 1 > 1/x + x$ for sufficiently large x; contradiction).

inf $A = 2$: Suppose $a + 1/a < 2$. Then $a^2 - 2a + 1 < 0$, which is not satisfied by any $a \in A$.
So $\inf A \geq 2$. But $2 \in A$ (take $1 + 1/1$).
So $\inf A = 2 = \min A$.

B: $\sup B = 5 + 1/10$.
$\inf B = \sqrt{2}$.

C: unbounded above, $\inf C = \min C = -\exp(-1)$.

4. Suppose $a+b \in A+B$ is arbitrary. Then $a \leq \sup A$ and $b \leq \sup B$, so $a+b \leq \sup(A)+\sup(B)$.
Then $\sup(A+B) \leq \sup A + \sup B$.
Suppose $\sup(A+B) < \sup A + \sup B$. Then $\exists x \in \mathbb{R}$ such that $\sup(A+B) < x < \sup A + \sup B$.
The first inequality implies $\forall (a+b) \in A+B$, $x > a+b$;
the second implies that $x < a + \sup B$ $(\forall a \in A)$
$\leq a + b$ $(\forall b \in B)$.
So $\forall a,b \in A,B$, $x > a+b$ and $x < a+b$ (contradiction).

$\therefore \sup(A+B) = \sup A + \sup B$.

5. Let $x<y \in \mathbb{R}$; we want $z \in \mathbb{R} \setminus \mathbb{Q}$, s.t. $x<z<y$.

By the Archimedean property, $\exists \eta \in \mathbb{Q}$ such that $x\sqrt{2} < \eta < y\sqrt{2}$; then $\eta/\sqrt{2}$ is irrational (Rudin, ch. 1, exercise 1). Thus, given the inequality $x < \eta/\sqrt{2} < y$, take $z = \eta/\sqrt{2}$.

6. • 2 and ½ are rational, $2^{1/2} = \sqrt{2} \notin \mathbb{Q}$.

• Yes: – 2 cases. Case I: $(\sqrt{2})^{(\sqrt{2})}$ is rational.
(Since $\sqrt{2} \notin \mathbb{Q}$, we are done.)
Case II: $(\sqrt{2})^{(\sqrt{2})} \notin \mathbb{Q}$.
Then $\left[(\sqrt{2})^{(\sqrt{2})}\right]^{(\sqrt{2})} = (\sqrt{2})^2 = 2 \in \mathbb{Q}$.

7. Clearly $A \subseteq \mathbb{Q}$ has an inf iff $A \subseteq \mathbb{R}$ has an inf that is rational. ~~Now a inf A iff a is a LB for A, and there exists $(a_n) \subset A$ s.t. $a_n \to a$.~~

8. Field: Clearly satisfies add. and mul. axioms; only non-trivial one is existence of inverse: if $a+b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$, then $\frac{a-b\sqrt{2}}{a^2-2b^2} = (a+b\sqrt{2})^{-1}$.
Contains $\mathbb{Q}$: obvious. Contains $\sqrt{2}$: by definition.

$\mathbb{Q}[\sqrt{2}]$ does not satisfy LUB property. E.g. $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$.
Proof: Suppose $\sqrt{3} = a+b\sqrt{2}$ for $a,b \in \mathbb{Q}$. Clearly $b \neq 0$ since $\sqrt{3} \notin \mathbb{Q}$.
Also, $a \neq 0$; otherwise $3 = (0+b\sqrt{2})^2 = 2b^2$; but ~~the power of~~ $b = p/q$ where p,q coprime; so $3q^2 = 2p^2$; power of 3 is odd on left but even on right (contradiction).
So $ab \neq 0$. In particular, $3 = (a+b\sqrt{2})^2 = a^2 + 2ab\sqrt{2} + 2b^2 \Rightarrow \sqrt{2} = \frac{3-a^2-2b^2}{2ab} \in \mathbb{Q}$
contradiction.

9. No: any subfield of $\mathbb{Q}$ ~~other than the same~~ will contain 1;
hence will contain $\underbrace{1+1+\dots+1}_{n \text{ times}} = n \in \mathbb{N}$, so contains $\mathbb{N}$,
then must include all inverses, and so includes $-n$ for all $n \in \mathbb{N}$,
and $1/n$ for all $n \in \mathbb{N}$; so any subfield of $\mathbb{Q}$ contains $\mathbb{Q}$.

10. Want $nr > s$. If $r, s \in \mathbb{Q}$, $n > s/r \in \mathbb{Q}$.
Write $r = a/b$, $s = c/d$; so want $n > \frac{ad}{bc}$.
If $ad < bc$ then $n = 17$ works, otherwise take $n = ad + 17$.

11. Given $k > 0$ and $x \in \mathbb{R}^k$, take $u = \frac{1}{|x|} \cdot x$ if $x \neq 0$,
and $u = (1, 0, \dots, 0)$ otherwise.